

OPTIMAL HARDY-SOBOLEV INEQUALITIES ON COMPACT RIEMANNIAN MANIFOLDS

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ABSTRACT. Given a compact Riemannian Manifold (M, g) of dimension $n \geq 3$, a point $x_0 \in M$ and $s \in (0, 2)$, the Hardy-Sobolev embedding yields the existence of $A, B > 0$ such that

$$(1) \quad \left(\int_M \frac{|u|^{\frac{2(n-s)}{n-2}}}{d_g(x, x_0)^s} dv_g \right)^{\frac{n-2}{n-s}} \leq A \int_M |\nabla u|_g^2 dv_g + B \int_M u^2 dv_g$$

for all $u \in H_1^2(M)$. It has been proved in Jaber [20] that $A \leq K(n, s)$ and that one can take any value $A > K(n, s)$ in (1), where $K(n, s)$ is the best possible constant in the Euclidean Hardy-Sobolev inequality. In the present manuscript, we prove that one can take $A = K(n, s)$ in (1).

Let (M, g) be a smooth compact Riemannian Manifold of dimension $n \geq 3$ without boundary, d_g be the Riemannian distance on M and $H_1^2(M)$ be the Sobolev space defined as the completion of $C^\infty(M)$ for the norm $u \mapsto \|u\|_2 + \|\nabla u\|_2$. We fix $x_0 \in M, s \in (0, 2)$ and let $2^*(s) := \frac{2(n-s)}{n-2}$ be the critical Hardy-Sobolev exponent. We endow the weighted Lebesgue space $L^p(M, d_g(\cdot, x_0)^{-s})$ with its natural norm $u \mapsto \|u\|_{p,s} := \left(\int_M |u|^p d_g(\cdot, x_0)^{-s} dv_g \right)^{\frac{1}{p}}$. It follows from the Hardy-Sobolev inequality that the Sobolev space $H_1^2(M)$ is continuously embedded in the weighted Lebesgue space $L^p(M, d_g(\cdot, x_0)^{-s})$ if and only if $1 \leq p \leq 2^*(s)$, and that this embedding is compact if and only if $1 \leq p < 2^*(s)$. From the embedding of $H_1^2(M)$ in $L^{2^*(s)}(M, d_g(\cdot, x_0)^{-s})$, one gets the existence of $A, B > 0$ such that

$$(2) \quad \|u\|_{2^*(s),s}^2 \leq A \|\nabla u\|_2^2 + B \|u\|_2^2$$

for all $u \in H_1^2(M)$. We let $K(n, s)$ be the optimal constant of the Euclidean Hardy-Sobolev inequality, that is

$$(3) \quad K(n, s)^{-1} := \inf_{\varphi \in C_c^\infty(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla \varphi|^2 dX}{\left(\int_{\mathbb{R}^n} \frac{|\varphi|^{2^*(s)}}{|X|^s} dX \right)^{\frac{2}{2^*(s)}}}$$

we denote by $C_c^\infty(\mathbb{R}^n)$ the set of C^∞ -smooth functions on \mathbb{R}^n with compact support and by $|\cdot|$ the Euclidean norm in \mathbb{R}^n . The value of $K(n, s)$ is

$$K(n, s) = [(n-2)(n-s)]^{-1} \left(\frac{1}{2-s} \omega_{n-1} \frac{\Gamma^2(n-s/2-s)}{\Gamma(2(n-s)/2-s)} \right)^{-\frac{2-s}{n-s}},$$

where ω_{n-1} is the volume of the unit sphere on \mathbb{R}^n and Γ is the Euler function. It was computed independently by Aubin [1], Rodemich [25] and Talenti [26] for the case $s = 0$, and the value for $s \in (0, 2)$ has been computed by Lieb (Theorem 4.3

in [22]). Following Hebey [16], we define $A_0(M, g, s)$ to be the best first constant of the Riemannian Hardy-Sobolev inequality, that is

$$(4) \quad A_0(M, g, x_0, s) := \inf\{A > 0 \exists B > 0 \text{ such that (2) holds for all } u \in H_1^2(M)\}.$$

For the Sobolev inequality ((2) when $s = 0$), Aubin proved in [1] that $A_0(M, g, x_0, 0) = K(n, 0)$. When $s \in (0, 2)$, the author proved in [20] that $A_0(M, g, x_0, s) = K(n, s)$. In particular, for any $\epsilon > 0$, there exists $B_\epsilon > 0$ such that we have :

$$(5) \quad \|u\|_{2^*(s), s}^2 \leq (K(n, s) + \epsilon) \int_M |\nabla u|_g^2 dv_g + B_\epsilon \int_M u^2 dv_g$$

for all $u \in H_1^2(M)$. See Thiam [28] for a version with an additional remainder. The constant B_ϵ obtained in [20] goes to $+\infty$ as $\epsilon \rightarrow 0$, and therefore the method used in [20] does not allow to take $\epsilon = 0$ in (5).

A natural question is to know whether the infimum $A_0(M, g, x_0, s)$ is achieved or not, that is if there exists $B > 0$ such that inequality (2) holds for all $u \in H_1^2(M)$ with $A = K(n, s)$. We prove the following :

Theorem 1. *Let (M, g) be a smooth compact Riemannian Manifold of dimension $n \geq 3$, $x_0 \in M$, and $s \in (0, 2)$. We let $2^*(s) := \frac{2(n-s)}{n-2}$ be the critical Hardy-Sobolev exponent. Then there exists $B_0(M, g, s, x_0) > 0$ depending on (M, g) and s such that*

$$(6) \quad \left(\int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq K(n, s) \int_M |\nabla u|_g^2 dv_g + B_0(M, g, s, x_0) \int_M u^2 dv_g$$

for all $u \in H_1^2(M)$.

When $s = 0$, Theorem 1 has been proved by Hebey-Vaugon [18] for best constant in the Sobolev embedding $H_1^2(M) \subset L^{2^*}(M)$, $2^* = 2n/(n-2)$. The best constant problem in the Sobolev embedding $H_1^p(M) \subset L^{p^*}(M)$, $p^* = pn/(n-p)$ ($n > p > 1$) has been studied by Druet [10] (see also Aubin-Li [2]) answering a conjecture of Aubin in [1]. The corresponding question for the embedding $H_2^2(M) \subset L^{2^\sharp}(M)$, $2^\sharp = 2n/(n-4)$ has been studied by Hebey [17], and for the Gagliardo-Nirenberg inequalities by Brouttelande [4] and Ceccon-Montenegro [7].

There is an important litterature about sharp constants for inequalities of Hardy-Sobolev type on domains of the Euclidean flat space \mathbb{R}^n . A general discussion is in the monograph [12] by Ghoussoub-Moradifam. Hardy-Sobolev inequalities are a subfamily of the Caffarelli-Kohn-Nirenberg inequalities (see [5]). The best constants and extremals for these inequalities on \mathbb{R}^n are well understood in the class of radially symmetric functions (see Catrina-Wang [6], Horushi [19] and Chou-Chu [8]). However, there are situations when extremals are not radially symmetrical as discovered by Catrina-Wang [6]. A historical survey on symmetry-breaking of the extremals is in Dolbeault-Esteban-Loss-Tarantello [9]. For Hardy-Sobolev-Maz'ya inequalities [23], we refer to Badiale-Tarantello [3], Musina [24] and Tertikas-Tintarev [27].

A last remark is that it follows from the analysis of the author in [20] that

$$(7) \quad \begin{cases} B_0(M, g, s, x_0) \geq K(n, s) \frac{(n-2)(6-s)}{12(2n-2-s)} \text{Scal}_g(x_0) & \text{if } n \geq 4 \\ \text{the Green's function's mass of } \Delta_g + \frac{B_0(M, g, s, x_0)}{K(3, s)} \text{ is nonpositive} & \text{if } n = 3, \end{cases}$$

where $\text{Scal}_g(x_0)$ is the scalar curvature at x_0 . The mass is defined at the end of Section 2.

The proof of Theorem 1 relies on the analysis of blowing-up families to critical non-linear elliptic equations. In Section 1, we prove a general convergence theorem for blowing-up solutions to Hardy-Sobolev equations. In Section 2, we prove Theorem 1 by adapting the arguments of Druet (in Druet [10]). We prove (7) in Section 2.

1. BLOW-UP AROUND x_0

We let (M, g) be a smooth compact Riemannian Manifold of dimension $n \geq 3$, $x_0 \in M$, $s \in (0, 2)$. We consider a family $(u_\alpha)_{\alpha>0}$ in $H_1^2(M)$, such that for all $\alpha > 0$, $u_\alpha \geq 0$, $u_\alpha \not\equiv 0$ and u_α is a solution to the problem

$$(8) \quad \begin{cases} \Delta_g u_\alpha + \alpha u_\alpha = \lambda_\alpha \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s} \\ \lambda_\alpha \in (0, K(n, s)^{-1}), \quad \|u_\alpha\|_{2^*(s), s} = 1. \end{cases}$$

Here $\Delta_g := -\text{div}_g(\nabla)$ is the Laplace-Beltrami operator. It follows from the regularity and the maximum principle of [20] that, for any $\alpha > 0$, $u_\alpha \in C^{0, \beta}(M) \cap C_{loc}^{2, \gamma}(M \setminus \{x_0\})$, $\beta \in (0, \min(1, 2-s))$, $\gamma \in (0, 1)$, and $u_\alpha > 0$. We define I_α as

$$(9) \quad v \in H_1^2(M) \setminus \{0\} \mapsto I_\alpha(v) := \frac{\int_M |\nabla v|_g^2 dv_g + \alpha \int_M v^2 dv_g}{\|u_\alpha\|_{2^*(s), s}^2}.$$

In particular, $I_\alpha(u_\alpha) = \int_M |\nabla u_\alpha|_g^2 dv_g + \alpha \int_M u_\alpha^2 dv_g = \lambda_\alpha$ for all $\alpha > 0$.

We claim that

$$(10) \quad u_\alpha \rightharpoonup 0 \text{ weakly in } H_1^2(M) \text{ as } \alpha \rightarrow +\infty.$$

We prove the claim. As one checked, $(u_\alpha)_{\alpha>0}$ is bounded in $H_1^2(M)$. Therefore, there exists $u^0 \in H_1^2(M)$ such that $u_\alpha \rightharpoonup u^0$ in $H_1^2(M)$ as $\alpha \rightarrow +\infty$. By $I_\alpha(u_\alpha) = \lambda_\alpha$, we get that $\|u_\alpha\|_2 \leq C_1 \alpha^{-1/2}$, where $C_1 > 0$ is independent of α . Since $H_1^2(M)$ is compactly embedded in $L^2(M)$, we then get that $\|u^0\|_2 = 0$. Hence $u^0 \equiv 0$. This proves the claim.

Since M is compact and $u_\alpha \in C^0(M)$ for all $\alpha > 0$, there exist $x_\alpha \in M$, $\mu_\alpha > 0$ such that

$$(11) \quad \max_M(u_\alpha) = u_\alpha(x_\alpha) = \mu_\alpha^{1-\frac{n}{2}}.$$

In the sequel, we denote by $\mathbb{B}_\rho(z) \subset M$ the geodesic ball of radius ρ centered at z .

Proposition 1. *We let $(u_\alpha)_{\alpha>0}$ be as in (9). Then $u_\alpha \rightarrow 0$ as $\alpha \rightarrow +\infty$ in $C_{loc}^0(M \setminus \{x_0\})$.*

Proof. We consider $y \in M \setminus \{x_0\}$, $\rho_y = \frac{1}{3}d_g(y, x_0)$. By (9), we have that $\Delta_g u_\alpha \leq F_\alpha u_\alpha$ in $\mathbb{B}_{2\rho_y}(y)$, where F_α is the function defined by $F_\alpha(x) = \lambda_\alpha u_\alpha^{2^*(s)-2}/d_g(\cdot, x_0)^s$. For any $r \in \left(\frac{n}{2}, \frac{n}{2-s}\right)$, we have that $\int_{\mathbb{B}_{2\rho_y}(y)} F_\alpha^r dv_g \leq C_2$ where $C_2 > 0$ is a constant independent of α . By Theorem 4.1 in Han-Lin [14] (see also Lemma 4 in the Appendix), it follows that there exists $C_3 = C_3(n, s, y, C_2) > 0$ independent of α such that, up to a subsequence, we have that

$$\max_{\mathbb{B}_{\rho_y}(y)} u_\alpha \leq C_3 \|u_\alpha\|_{L^2(\mathbb{B}_{2\rho_y}(y))}.$$

Since $u_\alpha \rightharpoonup 0$ in $H_1^2(M)$ as $\alpha \rightarrow +\infty$ then by the last inequality, we get

$$\lim_{\alpha \rightarrow +\infty} \|u_\alpha\|_{L^\infty(\mathbb{B}_{\rho_y}(y))} = 0.$$

Proposition 1 follows from a covering argument. \square

Proposition 2. *We let $(u_\alpha)_{\alpha>0}$ be as in (9). Then $\sup_M u_\alpha = +\infty$ as $\alpha \rightarrow +\infty$.*

Proof. We proceed by contradiction and assume that $\sup_M u_\alpha \not\rightarrow +\infty$ as $\alpha \rightarrow +\infty$. Then there exists $C_4 > 0$ independent of α such that $u_\alpha \leq C_4$. Since $u_\alpha \rightharpoonup 0$ as $\alpha \rightarrow +\infty$ in $H_1^2(M)$ then by dominated convergence Theorem we get that $\lim_{\alpha \rightarrow +\infty} \|u_\alpha\|_{2^*(s),s} = 0$. A contradiction since for all $\alpha > 0$, $\|u_\alpha\|_{2^*(s),s} = 1$. This ends the proof of Proposition 2. \square

Propositions 1 and 2 immediately yield the following:

Proposition 3. *We let $(u_\alpha)_{\alpha>0}$ be as in (9). Then $x_\alpha \rightarrow x_0$ as $\alpha \rightarrow +\infty$.*

In the sequel, we fix $R_0 \in (0, i_g(M))$, where $i_g(M) > 0$ is the injectivity radius of (M, g) . We fix $\eta_0 \in C_c^\infty(\mathbb{B}_{3R_0/4}(0) \subset \mathbb{R}^n)$ such that $\eta \equiv 1$ in $\mathbb{B}_{R_0/2}(0)$. The main result of this section is the following:

Theorem 2. *We let $(u_\alpha)_{\alpha>0}$ be as in (9). We consider a sequence $(z_\alpha)_{\alpha>0} \in M$ such that $\lim_{\alpha \rightarrow +\infty} z_\alpha = x_0$. We define the function \hat{u}_α on $\mathbb{B}_{R_0\mu_\alpha^{-1}}(0) \subset \mathbb{R}^n$ by*

$$(12) \quad \hat{u}_\alpha(X) = \mu_\alpha^{\frac{n}{2}-1} u_\alpha(\exp_{z_\alpha}(\mu_\alpha X)),$$

where $\exp_{z_\alpha}^{-1} : \Omega_\alpha \rightarrow \mathbb{B}_{R_0}(0)$ is the exponential map at z_α . We assume that

$$(13) \quad d_g(x_\alpha, z_\alpha) = O(\mu_\alpha) \text{ when } \alpha \rightarrow +\infty.$$

Then

$$(14) \quad d_g(z_\alpha, x_0) = O(\mu_\alpha) \text{ when } \alpha \rightarrow +\infty$$

and, up to a subsequence, $\eta_\alpha \hat{u}_\alpha$ converge to \hat{u} weakly in $D_1^2(\mathbb{R}^n)$ and uniformly in $C_{loc}^{0,\beta}(\mathbb{R}^n)$, for all $\beta \in (0, \min(1, 2-s))$, where $\eta_\alpha := \eta_0(\mu_\alpha \cdot)$ and

$$\hat{u}(X) = \left(\frac{a^{\frac{2-s}{2}} k^{\frac{2-s}{2}}}{a^{2-s} + |X - X_0|^{2-s}} \right)^{\frac{n-2}{2-s}} \text{ for all } X \in \mathbb{R}^n$$

with $X_0 \in \mathbb{R}^n$, $a > 0$ and $k^{2-s} = (n-2)(n-s)K(n, s)$. In particular \hat{u} verifies

$$(15) \quad \Delta_\delta \hat{u} = K(n, s)^{-1} \frac{\hat{u}^{2^*(s)-1}}{|X - X_0|^s} \text{ in } \mathbb{R}^n \text{ and } \int_{\mathbb{R}^n} \frac{\hat{u}^{2^*(s)}}{|X - X_0|^s} dX = 1,$$

where $|\cdot|$ is the Euclidean norm on \mathbb{R}^n and δ is the Euclidean metric of \mathbb{R}^n .

Proof of Theorem 2. We consider $(u_\alpha)_\alpha$, $(z_\alpha)_{\alpha>0} \in M$ and \hat{u}_α as in the statement of the theorem. We define the metric $\hat{g}_\alpha : X \mapsto \exp_{z_\alpha}^* g(\mu_\alpha X)$ and also on \mathbb{R}^n , the vectors $X_\alpha = \mu_\alpha^{-1} \exp_{z_\alpha}^{-1}(x_\alpha)$ and $X_{0,\alpha} = \mu_\alpha^{-1} \exp_{z_\alpha}^{-1}(x_0)$. It follows from (13) that

$$(16) \quad \frac{d_g(z_\alpha, x_\alpha)}{\mu_\alpha} = |X_\alpha| = O(1) \text{ when } \alpha \rightarrow +\infty.$$

The proof of Theorem 2 proceeds in several steps :

Step 1.0 : We claim that, for all $\alpha > 0$, \hat{u}_α verifies

$$(17) \quad \Delta_{\hat{g}_\alpha} \hat{u}_\alpha + \alpha \mu_\alpha^2 \hat{u}_\alpha = \lambda_\alpha \frac{\hat{u}_\alpha^{2^*(s)-1}}{d_{\hat{g}_\alpha}(X, X_{0,\alpha})^s}.$$

Proof. Indeed, we consider $\alpha > 0$, $X \in \mathbb{B}_{R_\alpha \mu_\alpha^{-1}}(0)$. Letting $x = \exp_{z_\alpha}(\mu_\alpha X)$, we then obtain

$$(18) \quad \Delta_{\hat{g}_\alpha} \hat{u}_\alpha(X) = \mu_\alpha^{\frac{n}{2}+1} \Delta_g u_\alpha(x).$$

and

$$(19) \quad \frac{\hat{u}_\alpha^{2^*(s)-1}}{d_{\hat{g}_\alpha}(X, X_{0,\alpha})^s} = \mu_\alpha^{(\frac{n}{2}-1)(2^*(s)-1)+s} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s}.$$

Since u_α verifies equation (9) then plugging (18) and (19) into (9), we get the claim. \square

Step 1.1: We claim that there exists $\hat{u} \in D_1^2(\mathbb{R}^n)$, $\hat{u} \not\equiv 0$ such that, up to a subsequence, $(\eta_\alpha \hat{u}_\alpha)_{\alpha>0}$ converge weakly to \hat{u} , as $\alpha \rightarrow +\infty$, in $D_1^2(\mathbb{R}^n)$.

Proof. Indeed, for all $\alpha > 0$, we can write :

$$(20) \quad \int_{\mathbb{R}^n} |\nabla(\eta_\alpha \hat{u}_\alpha)|_{\hat{g}_\alpha}^2 dv_{\hat{g}_\alpha} = \int_{\mathbb{R}^n} \eta_\alpha (\Delta_{\hat{g}_\alpha} \eta_\alpha) \hat{u}_\alpha^2 dv_{\hat{g}_\alpha} + \int_{\mathbb{R}^n} \eta_\alpha^2 |\nabla \hat{u}_\alpha|_{\hat{g}_\alpha}^2 dv_{\hat{g}_\alpha}$$

Since $\mu_\alpha \rightarrow 0$ as $\alpha \rightarrow +\infty$ then, up to a subsequence of $(\mu_\alpha)_{\alpha>0}$, there exists $C_5 > 0$ independent of α such that we have in the sense of bilinear form

$$(21) \quad C_5^{-1} \delta(X) \leq \hat{g}_\alpha(X) \leq C_5 \delta(X)$$

for all $X \in \mathbb{B}_{\frac{3R_0}{4\mu_\alpha}}(0)$ where δ is the Euclidean metric on \mathbb{R}^n . Relations (20) and (21) imply that there exists a constant $C_6 > 0$ independent of α such that

$$(22) \quad \int_{\mathbb{R}^n} |\eta_\alpha (\Delta_{\hat{g}_\alpha} \eta_\alpha)| \hat{u}_\alpha^2 dv_{\hat{g}_\alpha} \leq C_6 \mu_\alpha^2 \int_{\mathbb{B}_{\frac{3R_0}{4\mu_\alpha}}(0)} \hat{u}_\alpha^2 dv_{\hat{g}_\alpha},$$

By passing to $\mathbb{B}_{\frac{3R_0}{4}}(z_\alpha)$ in (22) via the exponential chart $(\Omega_\alpha, \exp_{z_\alpha}^{-1})$ (by taking $x = \exp_{z_\alpha}(\mu_\alpha X)$), we obtain that

$$(23) \quad \int_{\mathbb{R}^n} |\eta_\alpha (\Delta_{\hat{g}_\alpha} \eta_\alpha)| \hat{u}_\alpha^2 dv_{\hat{g}_\alpha} \leq C_7 \int_M u_\alpha^2 dv_g,$$

where $C_7 > 0$ is a constant independent of α . Since $\|u_\alpha\|_{H_1^2(M)} = O(1)$ when $\alpha \rightarrow +\infty$, then relation (23) yields

$$(24) \quad \int_{\mathbb{R}^n} |\nabla \eta_\alpha|_{\hat{g}_\alpha}^2 \hat{u}_\alpha^2 dv_{\hat{g}_\alpha} \leq C_8,$$

where $C_8 > 0$ is a constant independent of α . On the other hand, we write that

$$(25) \quad \begin{aligned} \int_{\mathbb{R}^n} \eta_\alpha^2 |\nabla \hat{u}_\alpha|_{\hat{g}_\alpha}^2 dv_{\hat{g}_\alpha} &\leq \int_{\mathbb{B}_{\frac{3R_0}{4}}(z_\alpha)} |\nabla u_\alpha|_g^2 dv_g \\ &\leq I_\alpha(u_\alpha) - \int_M \alpha u_\alpha^2 dv_g < K(n, s)^{-1} \end{aligned}$$

for all $\alpha > 0$. Plugging (24) and (25) into (20), we then obtain that

$$\int_{\mathbb{R}^n} |\nabla(\eta_\alpha \hat{u}_\alpha)|_{\hat{g}_\alpha}^2 dv_{\hat{g}_\alpha} \leq C_9$$

where $C_9 > 0$ is a constant independent of α . The last relation and (21) give

$$(26) \quad \int_{\mathbb{R}^n} |\nabla(\eta_\alpha \hat{u}_\alpha)|_\delta^2 dX \leq C_5^{n/2+1} \int_{\mathbb{R}^n} |\nabla(\eta_\alpha \hat{u}_\alpha)|_{\hat{g}_\alpha}^2 dv_{\hat{g}_\alpha} \leq C_{10}$$

where $C_{10} > 0$ is a constant independent of α . This implies that the sequence $(\eta_\alpha \hat{u}_\alpha)_{\alpha>0}$ is bounded in $D_1^2(\mathbb{R}^n)$ then there exists $\hat{u} \in D_1^2(\mathbb{R}^n)$ such that, up to a subsequence, $\eta_\alpha \hat{u}_\alpha \rightharpoonup \hat{u}$ as $\alpha \rightarrow +\infty$.

It remains to prove that $\hat{u} \not\equiv 0$. Indeed, since $\hat{u}_\alpha \leq 1$ and $\lambda_\alpha \in (0, K(n, s)^{-1})$ then for all $X \in \mathbb{B}_{R_\alpha \mu_\alpha^{-1}}(0)$, we can write :

$$(27) \quad \begin{aligned} \Delta_{\hat{g}_\alpha} \hat{u}_\alpha(X) &= \lambda_\alpha \frac{\hat{u}_\alpha^{2^*(s)-1}}{d_{\hat{g}_\alpha}(X, X_{0,\alpha})^s} - \alpha \mu_\alpha^2 \hat{u}_\alpha \\ &\leq \frac{K(n, s)^{-1}}{d_{\hat{g}_\alpha}(X, X_{0,\alpha})^s} \hat{u}_\alpha = F_\alpha(X) \hat{u}_\alpha, \end{aligned}$$

where

$$F_\alpha(X) := \frac{K(n, s)^{-1}}{d_{\hat{g}_\alpha}(X, X_{0,\alpha})^s}.$$

We consider $r \in (\frac{n}{2}, \frac{n}{s})$. It follows from (21) that

$$(28) \quad C_5^{-1/2} |X - X_{0,\alpha}| \leq d_{\hat{g}_\alpha}(X, X_{0,\alpha}) \leq C_5^{1/2} |X - X_{0,\alpha}|.$$

and

$$(29) \quad C_5^{-n/2} \leq \sqrt{\det(\hat{g}_\alpha)}(X) \leq C_5^{n/2}$$

for all $X \in \mathbb{B}_{R_0}(0)$. We distinguish two cases :

Case 1.1.1 : $X_{0,\alpha} \rightarrow X_0$ as $\alpha \rightarrow +\infty$. In this case, we get with (16) that for all $\alpha > 0$, $X_\alpha, X_{0,\alpha} \in \mathbb{B}_{R_1}(0)$, for $R_1 > 0$ sufficiently large and by (28) and (29), we obtain that $\int_{\mathbb{B}_{2R_1}(0)} F_\alpha^r dv_{\hat{g}_\alpha} \leq C_{11}$, where $C_{11} > 0$ is a constant independent of α .

Case 1.1.2 : $X_{0,\alpha} \rightarrow +\infty$ as $\alpha \rightarrow +\infty$. In this case, coming back to relations (28), (29) and by dominated convergence Theorem, we get that $\lim_{\alpha \rightarrow +\infty} \|F_\alpha\|_{C^0(\mathbb{B}_{2R_1}(0))} = 0$, where for all $\alpha > 0$, $X_\alpha \in \mathbb{B}_{R_1}(0)$. It follows that $\int_{\mathbb{B}_{R_1}(0)} F_\alpha^r dv_{\hat{g}_\alpha} \leq C_{12}$ where $C_{12} > 0$ is independent of α .

Hence, we get in both cases, up to a subsequence, that there exists $R_1 > 0$ such that for all $\alpha > 0$, $X_\alpha \in \mathbb{B}_{R_1}(0)$ and $\int_{\mathbb{B}_{2R_1}(0)} F_\alpha^r dv_{\hat{g}_\alpha} \leq C_{13}$, where $C_{13} = \max(C_{11}, C_{12})$. Moreover, for all $\theta \leq \min(1, 2-s)$, $\hat{u}_\alpha \in C^{0,\theta}(\mathbb{B}_{2R_1}(0))$ and $\hat{u}_\alpha > 0$. Thanks to Theorem 4.1 in Han-Lin [14] (see also Lemma 3 in the Appendix), we get that

$$(30) \quad 1 = \max_{\mathbb{B}_{R_1}(0)} \hat{u}_\alpha \leq C_{14} \|\hat{u}_\alpha\|_{L^2(\mathbb{B}_{2R_1}(0))},$$

where $C_{14} > 0$ is a constant independent of α . Since $(\eta_\alpha \hat{u}_\alpha)_{\alpha>0}$ is bounded and converges weakly to \hat{u} as $\alpha \rightarrow +\infty$ in $D_1^2(\mathbb{R}^n)$, the convergence is strong in L_{loc}^2 and then, letting $\alpha \rightarrow +\infty$ in (30), we get that $\|\hat{u}\|_{L^2(\mathbb{B}_{2R_1}(0))} \geq C_{14}^{-1}$ and then $\hat{u} \not\equiv 0$. This ends the proof of Step 1.1. \square

Step 1.2: We claim that $\lambda_\alpha \rightarrow K(n, s)^{-1}$ as $\alpha \rightarrow +\infty$.

Proof. Indeed, since for all $\alpha > 0$, we have $\lambda_\alpha \in (0, K(n, s)^{-1})$ then, up to a subsequence, $\lambda_\alpha \rightarrow \lambda \leq K(n, s)^{-1}$ as $\alpha \rightarrow +\infty$. We proceed by contradiction and assume that $\lambda \neq K(n, s)^{-1}$. Then there exist $\epsilon_0 > 0$ and $\alpha_0 > 0$ such that for all $\alpha > \alpha_0$:

$$(31) \quad K(n, s)^{-1} > \lambda + \epsilon_0.$$

By Jaber [20], for all $\epsilon > 0$ there exist $B_\epsilon > 0$ such that for all $\alpha > 0$, we have :

$$(32) \quad \left(\int_M \frac{|u_\alpha|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq (K(n, s) + \epsilon) \int_M |\nabla u_\alpha|_g^2 dv_g + B_\epsilon \int_M u_\alpha^2 dv_g.$$

Since $\|u_\alpha\|_{2^*(s), s} = 1$, $I_\alpha(u_\alpha) = \lambda_\alpha$ and $u_\alpha \rightarrow 0$ in $L^2(M)$ as $\alpha \rightarrow +\infty$ then

$$1 \leq \left(\frac{1}{\lambda_\alpha + \epsilon_0} + \epsilon \right) \lambda_\alpha + o(1).$$

Letting $\alpha \rightarrow +\infty$ and then $\epsilon \rightarrow 0$ in the last relation, we obtain that $\frac{\lambda}{\lambda + \epsilon_0} \geq 1$, a contradiction since $\lambda \geq 0$ and $\epsilon_0 > 0$. This proves that $\lambda = K(n, s)^{-1}$. \square

Step 1.3: We claim that there exists $A \geq 0$ such that \hat{u} verifies on $C_c^\infty(\mathbb{R}^n)$:

$$(33) \quad \Delta_\delta \hat{u} + A\hat{u} = \begin{cases} K(n, s)^{-1} \frac{\hat{u}^{2^*(s)-1}}{|X - X_0|^s} & \text{if } X_{0, \alpha} \xrightarrow{\alpha \rightarrow +\infty} X_0 \\ 0 & \text{if } |X_{0, \alpha}| \xrightarrow{\alpha \rightarrow +\infty} +\infty. \end{cases}$$

Proof. We consider $R > 0$ and $\varphi \in C^\infty(\mathbb{B}_R(0))$. Indeed, thanks to Cartan's expansion of the metric g (see for instance [21]), we have for all $\alpha > 0$:

$$\hat{g}_\alpha(X) = \delta(z_\alpha) + o(\mu_\alpha)$$

uniformly on $\mathbb{B}_R(0)$. This implies that

$$(34) \quad \int_{\mathbb{R}^n} \langle \nabla \hat{u}_\alpha, \nabla \varphi \rangle_{\hat{g}_\alpha} dv_{\hat{g}_\alpha} = \int_{\mathbb{B}_R(0)} \langle \nabla \hat{u}_\alpha, \nabla \varphi \rangle_\delta dX + o(\mu_\alpha)$$

Since $\eta_\alpha \hat{u}_\alpha \rightharpoonup \hat{u}$ on $D_1^2(\mathbb{R}^n)$ and $\mu_\alpha \rightarrow 0$ as $\alpha \rightarrow +\infty$ then by (34), we get that

$$(35) \quad \lim_{\alpha \rightarrow +\infty} \int_{\mathbb{R}^n} \langle \nabla(\eta_\alpha \hat{u}_\alpha), \nabla \varphi \rangle_{\hat{g}_\alpha} dv_{\hat{g}_\alpha} = \int_{\mathbb{B}_R(0)} \langle \nabla \hat{u}, \nabla \varphi \rangle_\delta dX.$$

Now, since $I_\alpha(u_\alpha) = \lambda_\alpha$ and $\lambda_\alpha \in (0, K(n, s)^{-1})$ then we get

$$(36) \quad \alpha \mu_\alpha^2 \int_{\mathbb{B}_R(0)} \hat{u}_\alpha^2 dv_{\hat{g}_\alpha} < K(n, s)^{-1}.$$

By dominated convergence Theorem, we obtain that

$$(37) \quad \int_{\mathbb{B}_R(0)} \hat{u}_\alpha^2 dv_{\hat{g}_\alpha} \xrightarrow{\alpha \rightarrow +\infty} \int_{\mathbb{B}_R(0)} \hat{u}^2 dX.$$

Together, Relations (36) and (37) give that

$$\alpha \mu_\alpha^2 \leq \frac{K(n, s)^{-1}}{\int_{\mathbb{B}_R(0)} \hat{u}^2 dX} + o(1).$$

Hence, $\alpha \mu_\alpha^2 = O(1)$ and there exists $A \geq 0$ such that, up to a subsequence, $\lim_{\alpha \rightarrow +\infty} \alpha \mu_\alpha^2 = A$. Using dominated convergence Theorem again, we obtain that

$$(38) \quad \lim_{\alpha \rightarrow +\infty} \int_{\mathbb{R}^n} \alpha \mu_\alpha^2 \hat{u}_\alpha \varphi dv_{\hat{g}_\alpha} = A \int_{\mathbb{R}^n} \hat{u} \varphi dX.$$

At last, we consider the sequence $(h_\alpha)_{\alpha>0}$ defined on $\mathbb{B}_R(0)$ by :

$$X \in \mathbb{B}_R(0) \mapsto h_\alpha(X) = \frac{\hat{u}_\alpha^{2^*(s)-1} \varphi}{d_{\hat{g}_\alpha}(X, X_{0,\alpha})^s} \sqrt{\det(\hat{g}_\alpha)}.$$

We claim that

$$\lim_{\alpha \rightarrow +\infty} \int_{\mathbb{B}_R(0)} h_\alpha dX = \begin{cases} 0 & \text{if } |X_{0,\alpha}| \xrightarrow{\alpha \rightarrow +\infty} +\infty \\ \int_{\mathbb{B}_R(0)} \frac{\varphi \hat{u}^{2^*(s)-1} dX}{|X - X_0|^s} & \text{if } X_{0,\alpha} \xrightarrow{\alpha \rightarrow +\infty} X_0. \end{cases}$$

We distinguish two cases :

Case 1.3.1 : $X_{0,\alpha} \rightarrow +\infty$ as $\alpha \rightarrow +\infty$. In this case, $\lim_{\alpha \rightarrow +\infty} d_{\hat{g}_\alpha}(X, X_{0,\alpha})^{-s} = 0$ in $C_c^0(\mathbb{R}^n)$. Hence $\lim_{\alpha \rightarrow +\infty} \int_{\mathbb{B}_R(0)} h_\alpha dX = 0$. This proves the claim in case 1.3.1.

Case 1.3.2 : There exists $X_0 \in \mathbb{R}^n$ such that $X_{0,\alpha} \rightarrow X_0$ as $\alpha \rightarrow +\infty$. Let us consider the function h defined on $\mathbb{B}_R(0)$ by $X \mapsto h(X) = (\hat{u}^{2^*(s)-1} \varphi)(X) / |X - X_0|^s$. In this case, for all $\epsilon > 0$, there exists $\alpha_1 = \alpha_1(\epsilon) > 0$ such that for all $\alpha > \alpha_1$, $X_{0,\alpha} \in \mathbb{B}_{\frac{\epsilon}{2}}(X_0)$. Then for all $X \in \mathbb{B}_R(0) \setminus \mathbb{B}_\epsilon(X_0)$, we have : $|X - X_{0,\alpha}| \geq \frac{\epsilon}{2}$. This implies that there exists a constant $C_{15} = C_{15}(\epsilon) > 0$ independent of α such that $|h_\alpha| \leq C_{15} \cdot |\varphi|$. Coming back to dominated convergence Theorem, we obtain with the last relation that

$$(39) \quad \lim_{\alpha \rightarrow +\infty} \int_{\mathbb{B}_R(0) \setminus \mathbb{B}_\epsilon(X_0)} h_\alpha(X) dX = \int_{\mathbb{B}_R(0) \setminus \mathbb{B}_\epsilon(X_0)} h(X) dX.$$

On the other hand, we get by (28) and (29) that

$$(40) \quad \left| \int_{\mathbb{B}_\epsilon(X_0)} h_\alpha dX \right| \leq C_{10} \|\varphi\|_\infty \int_{\mathbb{B}_{2\epsilon}(X_{0,\alpha})} \frac{dX}{|X - X_{0,\alpha}|^s} \leq C_{16} \cdot \epsilon^{n-s}.$$

where $C_{16} > 0$ is a constant independent of α . In a similar way, we prove that

$$(41) \quad \left| \int_{\mathbb{B}_\epsilon(X_0)} h dX \right| \leq C_{17} \epsilon^{n-s},$$

where $C_{17} > 0$ is a constant independent of α . Combining (39), (40) and (41), it follows for all $\alpha > \alpha_1$ that

$$\left| \int_{\mathbb{B}_R(0)} h_\alpha dX - \int_{\mathbb{B}_R(0)} h dX \right| = o_\alpha(1) + O(\epsilon^{n-s}).$$

Letting $\alpha \rightarrow +\infty$ then $\epsilon \rightarrow 0$ in the last relation, we obtain that

$$(42) \quad \lim_{\alpha \rightarrow +\infty} \int_{\mathbb{R}^n} \frac{\hat{u}_\alpha^{2^*(s)-1} \varphi}{d_{\hat{g}_\alpha}(X, X_0)^s} dv_{\hat{g}_\alpha} = \int_{\mathbb{R}^n} \frac{\hat{u}^{2^*(s)-1} \varphi}{|X - X_0|^s} dX.$$

This proves the claim in case 1.3.2. Hence, by combining relations (35), (38) and (42) with (17), we get (33). This ends Step 1.3. \square

Step 1.4: We claim that $X_{0,\alpha} = \mu_\alpha^{-1} \exp_{z_\alpha}^{-1}(x_0)$ is bounded when $\alpha \rightarrow +\infty$.

Proof. We proceed by contradiction and we assume that $|X_{0,\alpha}| \rightarrow +\infty$ as $\alpha \rightarrow +\infty$. We proved in Step 1.3 that we obtain in this case :

$$(43) \quad \Delta_\delta \hat{u} + A \hat{u} = 0,$$

weakly on $C_c^\infty(\mathbb{R}^n)$. Let $\hat{\eta} \in C^\infty(\mathbb{R}^n)$ be such that $\hat{\eta} \equiv 1$ in $\mathbb{B}_1(0)$, $0 \leq \hat{\eta} \leq 1$ and $\hat{\eta} \equiv 0$ in $\mathbb{R}^n \setminus \mathbb{B}_2(0)$. Now, we consider $R > 0$ and define the function $\hat{\eta}_R$ on \mathbb{R}^n by $\hat{\eta}_R(X) = \hat{\eta}(R^{-1}X)$. Multiplying (43) by $\hat{\eta}_R \hat{u}$ and integrating by parts, we get that

$$(44) \quad \int_{\mathbb{R}^n} (\nabla \hat{u}, \nabla(\hat{\eta}_R \hat{u}))_\delta dX + A \int_{\mathbb{R}^n} \hat{\eta}_R \hat{u}^2 = 0.$$

To get the contradiction, we need the following lemma :

Lemma 1. *We claim that*

$$\lim_{R \rightarrow +\infty} \int_{\mathbb{R}^n} (\nabla \hat{u}, \nabla(\hat{\eta}_R \hat{u}))_\delta dX = \|\hat{u}\|_{D_1^2(\mathbb{R}^n)}^2.$$

Proof of Lemma 1: Indeed, we have that :

$$(45) \quad \int_{\mathbb{R}^n} (\nabla \hat{u}, \nabla(\hat{\eta}_R \hat{u}))_\delta dX = \int_{\mathbb{R}^n} \hat{\eta}_R |\nabla \hat{u}|_\delta^2 dX + \int_{\mathbb{R}^n} (\nabla \hat{u}, \nabla(\hat{\eta}_R) \hat{u})_\delta dX.$$

Applying dominated convergence Theorem, we get that

$$(46) \quad \lim_{R \rightarrow +\infty} \int_{\mathbb{R}^n} \hat{\eta}_R |\nabla \hat{u}|_\delta^2 dX = \|\hat{u}\|_{D_1^2(\mathbb{R}^n)}^2.$$

On the other hand, we obtain by Inequalities of Cauchy-Schwarz then by Hölder's inequalities that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} (\nabla \hat{u}, \nabla(\hat{\eta}_R) \hat{u})_\delta dX \right| &\leq \|\hat{u}\|_{D_1^2(\mathbb{R}^n)}^2 \times \frac{C_{18}}{R^2} \int_{\mathbb{B}_{2R}(0) \setminus \mathbb{B}_R(0)} \hat{u}^2 dX \\ &\leq C_{19} \|\hat{u}\|_{D_1^2(\mathbb{R}^n)}^2 \times \left(\int_{\mathbb{B}_{2R}(0) \setminus \mathbb{B}_R(0)} \hat{u}^{2^*} dX \right)^{\frac{2}{2^*}}, \end{aligned}$$

where $2^* = 2n/(n-2)$ and $C_{18}, C_{19} > 0$ are independents of α . It follows from the last relation, Sobolev's embedding theorem and the dominated convergence theorem that

$$(47) \quad \lim_{R \rightarrow +\infty} \int_{\mathbb{R}^n} (\nabla \hat{u}, \nabla(\hat{\eta}_R) \hat{u})_\delta dX = 0.$$

Letting $R \rightarrow +\infty$ in (45) and thanks to relations (46) and (47), we get the claim. This proves Lemma 1.

Now, going back to relation (44) and thanks to Lemma 1, we get that

$$\|\hat{u}\|_{D_1^2(\mathbb{R}^n)}^2 + A \int_{\mathbb{R}^n} \hat{\eta}_R \hat{u}^2 = o_R(1),$$

where $\lim_{R \rightarrow +\infty} o_R(1) = 0$. Thus is a contradiction since $\hat{\eta}_R \hat{u}^2 \geq 0$ and $\hat{u} \not\equiv 0$. This contradiction completes the proof of Step 1.4. \square

As a consequence, Step 1.4 implies that $|X_{0,\alpha}| = O(1)$ when $\alpha \rightarrow +\infty$, which yields (14). Therefore, there exists $X_0 \in \mathbb{R}^n$ such that the function \hat{u} verifies in the distribution sense :

$$(48) \quad \Delta_\delta \hat{u} + A \hat{u} = K(n, s)^{-1} \frac{\hat{u}^{2^*(s)-1}}{|X - X_0|^s}.$$

Step 1.5: We claim that $A = 0$.

Proof. We proceed by contradiction and assume that $A > 0$. At first, let us prove that $\hat{u} \in L^2(\mathbb{R}^n)$. Multiplying (48) by $\hat{\eta}_R \hat{u}$ and integrating over \mathbb{R}^n , we obtain

$$(49) \quad \int_{\mathbb{R}^n} (\nabla \hat{u}, \nabla (\hat{\eta}_R \hat{u}))_\delta dX + A \int_{\mathbb{R}^n} \hat{\eta}_R \hat{u}^2 dX = K(n, s)^{-1} \int_{\mathbb{R}^n} \hat{\eta}_R \frac{\hat{u}^{2^*(s)}}{|X - X_0|^s} dX.$$

We claim that $\hat{u}^{2^*(s)} |X - X_0|^{-s} \in L^1(\mathbb{R}^n)$. We prove the claim. For all $\alpha > 0$, we have that $\int_M \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g = 1$. Then for $R > 0$, we obtain by a change of variable that $\int_{\mathbb{B}_R(0)} \frac{|\eta_\alpha \hat{u}_\alpha|^{2^*(s)}}{d_{\hat{g}_\alpha}(X, X_{0, \alpha})^s} dv_{\hat{g}_\alpha} \leq 1$. Letting $\alpha \rightarrow +\infty$ then $R \rightarrow +\infty$, we get that

$$(50) \quad \int_{\mathbb{R}^n} \frac{\hat{u}^{2^*(s)}}{|X - X_0|^s} dX \leq 1.$$

This proves the claim.

Letting $R \rightarrow +\infty$ in (49) and using (50), we get, thanks to Lemma 1, that $\lim_{R \rightarrow +\infty} A \int_{\mathbb{R}^n} \hat{\eta}_R \hat{u}^2 dX \leq C_{20}$, where $C_{20} > 0$ is independent of α . Applying Beppo-Livi Theorem in the last relation, we get that $\hat{u}^2 \in L^1(\mathbb{R}^n)$. Now, we consider the function

$$\begin{aligned} f : \mathbb{R}^n \setminus \{X_0\} \times \mathbb{R} &\rightarrow \mathbb{R} \\ (X, v) &\mapsto f(X, v) = \left(\frac{K(n, s)^{-1} |v|^{2^*(s)-2}}{|X - X_0|^s} - A \right) v. \end{aligned}$$

f is clearly continuous on $\mathbb{R}^n \setminus \{X_0\} \times \mathbb{R}$ and \hat{u} verifies $\Delta_\delta \hat{u} = f(X, \hat{u})$, it follows by standard elliptic theory (see for instance [13]) that $\hat{u} \in C_c^\infty(\mathbb{R}^n \setminus \{X_0\}) \cap H_{2, loc}^1(\mathbb{R}^n \setminus \{X_0\})$. By the Claim 5.3 in [11] (once one checks that \hat{u} and f satisfy all the condition of the Claim), we obtain after simple computations that $A \int_{\mathbb{R}^n} \hat{u}^2 dX = 0$. A contradiction since $\hat{u} \in L^2(\mathbb{R}^n)$ and $\hat{u} \not\equiv 0$. This ends the proof of Step 1.5. \square

As a consequence, Step 1.6 implies that there exists $X_0 \in \mathbb{R}^n$ such that the function \hat{u} verifies in the distribution sense :

$$(51) \quad \Delta_\delta \hat{u} = K(n, s)^{-1} \frac{\hat{u}^{2^*(s)-1}}{|X - X_0|^s}.$$

Step 1.6 : We claim that there exists $a > 0$ such that

$$(52) \quad \hat{u}(X) = \left(\frac{a^{\frac{2-s}{2}} k^{\frac{2-s}{2}}}{a^{2-s} + |X - X_0|^{2-s}} \right)^{\frac{n-2}{2-s}} \text{ for all } X \in \mathbb{R}^n,$$

where $k^{2-s} := (n-s)(n-2)K(n, s)$.

Proof. Indeed, Multiplying (51) by $\hat{\eta}_R \hat{u}$, integrating over \mathbb{R}^n and letting $R \rightarrow +\infty$, we obtain that

$$\int_{\mathbb{R}^n} |\nabla \hat{u}|^2 dX = K(n, s)^{-1} \int_{\mathbb{R}^n} \frac{\hat{u}^{2^*(s)}}{|X - X_0|^s} dX.$$

Thanks to the definition of $K(n, s)$ and with the last relation, we get that

$$(53) \quad \int_{\mathbb{R}^n} \frac{\hat{u}^{2^*(s)}}{|X - X_0|^s} dX \geq 1.$$

Inequalities (53) and (50) give that

$$\int_{\mathbb{R}^n} \frac{\hat{u}^{2^*(s)}}{|X - X_0|^s} = 1.$$

This implies that, up to a translation, \hat{u} is a minimizer for the Euclidean Hardy-Sobolev inequality. By Lemma 3 in [9] (see Chou-Chu [8], Horiuchi [19] and also Theorem 4.3 in Lieb [22] and Theorem 4 in Catrina-Wang [6]), we get that $\hat{u}(X) = b(c + |X - X_0|^{2-s})^{-\frac{n-2}{2-s}}$ for some $b \neq 0$ and $c > 0$. Since \hat{u} satisfies (51), we get (52). This proves the claim. \square

Step 1.7 : We claim that, up to a subsequence, $\eta_\alpha \hat{u}_\alpha \rightarrow \hat{u}$ in $C_{loc}^{0,\beta}(\mathbb{R}^n)$, for all $\beta \in (0, \min(1, 2-s))$,

Proof. Given $R' > 0$, we get by Step 1.0 (equation (17)) and Step 1.6, up to a subsequence of $(\hat{u}_\alpha)_{\alpha>0}$, that $\Delta_{\hat{g}_\alpha} \hat{u}_\alpha = F_\alpha$ on $C^\infty(\mathbb{B}_R(0))$ where $F_\alpha(X) = -\alpha \mu_\alpha^2 \hat{u}_\alpha + \lambda_\alpha \frac{\hat{u}_\alpha^{2^*(s)-1}}{d_{\hat{g}_\alpha}(X, X_{0,\alpha})^s}$. We consider $p \in (n/2, \inf(n/s, n))$. Knowing that $\hat{u}_\alpha \leq 1$ leads to $\|F_\alpha\|_{L^p(\mathbb{B}_{R'}(0))} = O(1)$.

It follows by standard elliptic theory (see [13]) that for all $\beta \in (0, \min(1, 2-s))$ and all $R < R'$, $(\hat{u}_\alpha) \in C^{0,\beta}(\mathbb{B}_R(0))$ and there exists $C_{21} = C_{21}(M, g, s, R, R', \beta) > 0$ such that $\|\hat{u}_\alpha\|_{C^{0,\beta}(\mathbb{B}_R(0))} \leq C$. Therefore the convergence holds in $C^{0,\beta'}(\mathbb{B}_R(0))$ for all $\beta' < \beta$. This ends the proof of Step 1.7. \square

Theorem 2 follows from Steps 1.0 to 1.7.

Corollary 1. *Up to a subsequence of $(x_\alpha)_{\alpha>0}$, we have $d_g(x_0, x_\alpha) = o(\mu_\alpha)$ when $\alpha \rightarrow +\infty$. Moreover, $(\eta_\alpha \hat{u}_\alpha)$ goes weakly to \hat{u} in $D_1^2(\mathbb{R}^n)$ and strongly in $C_{loc}^{0,\beta}(\mathbb{R}^n)$ for $\beta \in (0, \inf\{1, 2-s\})$ where $\hat{u}(X) = \left(\frac{k^{2-s}}{k^{2-s} + |X|^{2-s}}\right)^{\frac{n-2}{2-s}}$ for all $X \in \mathbb{R}^n$ with $k^{2-s} = (n-2)(n-s)K(n, s)$. In addition,*

$$\lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{M \setminus \mathbb{B}_{R\mu_\alpha}(x_0)} \frac{\hat{u}_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g = 0.$$

Proof. At first, we apply Theorem 2 with $z_\alpha = x_\alpha$. In this case, we get that $\eta_\alpha \hat{u}_\alpha \rightarrow \hat{u}$ in $C_c^0(\mathbb{R}^n)$ as $\alpha \rightarrow +\infty$. This implies that $\lim_{\alpha \rightarrow +\infty} \hat{u}_\alpha(0) = \hat{u}(0)$, but $\hat{u}_\alpha(0) = 1$ then $\hat{u}(0) = 1$.

Since $\hat{u}(0) = 1$ and $\|\hat{u}\|_\infty = 1$, Then 0 is a maximum of \hat{u} . On the other hand, we can see from the explicit form of \hat{u} in Theorem 2 that for all $X \in \mathbb{R}^n$, $\hat{u}(X) \leq \hat{u}(X_0)$. Therefore $X_0 = 0$. Hence, we obtain, up to a subsequence of $(z_\alpha)_{\alpha>0}$, that

$$(54) \quad d_g(x_\alpha, x_0) = \mu_\alpha d_{\hat{g}_\alpha}(X_{0,\alpha}, 0) = \mu_\alpha |X_{0,\alpha}| = o(\mu_\alpha).$$

We now apply Theorem 2 with $z_\alpha = x_0$: this is possible due to (54). With the change of variable $X = \mu_\alpha^{-1} \exp_{x_0}^{-1}(x)$, we write that

$$\int_{\mathbb{B}_{R\mu_\alpha}(x_0)} \frac{|u_\alpha|^{2^*(s)}}{d_g(x, x_0)^s} dv_g = \int_{\mathbb{B}_R(0)} \frac{|\hat{u}_\alpha|^{2^*(s)}}{|X|^s} dv_{\hat{g}_{0,\alpha}}$$

with $\hat{g}_{0,\alpha}(X) = \exp_{x_0}^* g(\mu_\alpha X)$, we get by applying the dominated convergence Theorem twice and thanks to Theorem 2 that

$$\begin{aligned} \lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{\mathbb{B}_{R\mu_\alpha}(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g &= \lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{\mathbb{B}_R(0)} \frac{|\hat{u}_\alpha|^{2^*(s)}}{|X|^s} dv_{\hat{g}_{0,\alpha}} \\ (55) \qquad \qquad \qquad &= \int_{\mathbb{R}^n} \frac{\hat{u}^{2^*(s)}}{|X|^s} dX = 1. \end{aligned}$$

Corollary 1 follows from this latest relation and $\|u_\alpha\|_{2^*(s),s} = 1$. \square

2. PROOF OF THEOREM 1

In order to prove Theorem 1, we proceed by contradiction and assume that for all $\alpha > 0$, there exists $\tilde{u}_\alpha \in H_1^2(M)$ such that

$$(56) \qquad \left(\int_M \frac{|\tilde{u}_\alpha|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} > K(n, s) \left(\int_M |\nabla \tilde{u}_\alpha|^2 dv_g + \alpha \int_M \tilde{u}_\alpha^2 dv_g \right).$$

We proceed in several steps :

Step 2.1: We claim that for all $\alpha > 0$ there exists $u_\alpha \in C^{0,\beta}(M) \cap C^{2,\theta}(M \setminus \{x_0\})$, $\beta \in (0, \min(1, 2-s))$, $\theta \in (0, 1)$ such that $u_\alpha > 0$ and verifies

$$(57) \qquad \Delta_g u_\alpha + \alpha u_\alpha = \lambda_\alpha \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s}$$

with $\lambda_\alpha \in (0, K(n, s)^{-1})$, $\lambda_\alpha = I_\alpha(u_\alpha)$ and $\int_M \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g = 1$.

Proof. Given $\alpha > 0$. By (56), there exists $\tilde{u}_\alpha \in H_1^2(M)$ that verifies $I_\alpha(\tilde{u}_\alpha) < K(n, s)^{-1}$. This implies that $\lambda_\alpha := \inf_{v \in H_1^2(M) \setminus \{0\}} I_\alpha(v) < K(n, s)^{-1}$. Hence, thanks to Jaber [20] (Theorem 4, see also Thiam [28]), we get the Claim of Step 2.1. \square

Step 2.2: Following Druet arguments in [10] (see also Hebey [17]), we claim that there exists $C_{22} > 0$ such that for all $x \in M$ et $\alpha > 0$, we have :

$$(58) \qquad d_g(x_0, x)^{\frac{n}{2}-1} u_\alpha(x) \leq C_{22}.$$

Proof. We proceed by contradiction and assume that there exists a sequence $(y_\alpha)_{\alpha>0} \in M$ such that

$$(59) \qquad \sup_{x \in M} d_g(x_0, x)^{\frac{n}{2}-1} u_\alpha(x) = d_g(x_0, y_\alpha)^{\frac{n}{2}-1} u_\alpha(y_\alpha)$$

and

$$(60) \qquad \lim_{\alpha \rightarrow +\infty} d_g(x_0, y_\alpha)^{\frac{n}{2}-1} u_\alpha(y_\alpha) = +\infty.$$

Since M is compact, we then obtain that $\lim_{\alpha \rightarrow +\infty} u_\alpha(y_\alpha) = +\infty$. Thanks to Proposition (1), we get that, up to a subsequence, $y_\alpha \rightarrow x_0$ as $\alpha \rightarrow \infty$. Now, for all $\alpha > 0$, we let $\hat{r}_\alpha = u_\alpha(y_\alpha)^{\frac{-2}{n-2}}$.

We claim that for a given $\alpha > 0$ and $R > 0$,

$$(61) \qquad \int_{\mathbb{B}_{\hat{r}_\alpha}(y_\alpha)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g = \varepsilon_R + o(1), \text{ where } \lim_{R \rightarrow +\infty} \varepsilon_R = 0.$$

Indeed, we fix $\rho > 0$. Since $y_\alpha \rightarrow x_0$ et $\hat{r}_\alpha \rightarrow 0$ as $\alpha \rightarrow +\infty$ then we write, up to a subsequence of $(y_\alpha)_{\alpha>0}$, that :

$$(62) \quad \int_{\mathbb{B}_{\hat{r}_\alpha}(y_\alpha)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g = \int_{\mathbb{B}_{\hat{r}_\alpha}(y_\alpha) \cap \mathbb{B}_\rho(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g.$$

Given $R > 0$. Thanks to Corollary 1, we have that

$$\int_{\mathbb{B}_\rho(x_0) \setminus \mathbb{B}_{R\mu_\alpha}(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g = \varepsilon_R + o(1),$$

where the function $\varepsilon_R : \mathbb{R} \rightarrow \mathbb{R}$ verifies $\lim_{R \rightarrow +\infty} \varepsilon_R = 0$. Therefore,

$$(63) \quad \begin{aligned} \int_{\mathbb{B}_{\hat{r}_\alpha}(y_\alpha)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g &= \int_{\mathbb{B}_{\hat{r}_\alpha}(y_\alpha) \cap \mathbb{B}_\rho(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g \\ &= \int_{\mathbb{B}_{\hat{r}_\alpha}(y_\alpha) \cap (\mathbb{B}_\rho(x_0) \setminus \mathbb{B}_{R\mu_\alpha}(x_0))} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g \\ &\quad + \int_{\mathbb{B}_{\hat{r}_\alpha}(y_\alpha) \cap \mathbb{B}_{R\mu_\alpha}(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g \\ &\leq \varepsilon_R + o(1) + \int_{\mathbb{B}_{\hat{r}_\alpha}(y_\alpha) \cap \mathbb{B}_{R\mu_\alpha}(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g, \end{aligned}$$

where the function $\varepsilon_R : \mathbb{R} \rightarrow \mathbb{R}$ verifies $\lim_{R \rightarrow +\infty} \varepsilon_R = 0$. We distinguish two cases :

Case 2.2.1 : $\mathbb{B}_{\hat{r}_\alpha}(y_\alpha) \cap \mathbb{B}_{R\mu_\alpha}(x_0) = \emptyset$. In this case, we obtain immediately (61) from (63).

Case 2.2.2 : $\mathbb{B}_{\hat{r}_\alpha}(y_\alpha) \cap \mathbb{B}_{R\mu_\alpha}(x_0) \neq \emptyset$. In this case, we obtain that

$$(64) \quad d_g(x_0, y_\alpha) \leq \hat{r}_\alpha + R\mu_\alpha.$$

By (60), we get that

$$(65) \quad \lim_{\alpha \rightarrow +\infty} \frac{\hat{r}_\alpha}{d_g(x_0, y_\alpha)} = 0.$$

Together, relations (64) and (65) give that

$$(66) \quad \frac{\hat{r}_\alpha}{\mu_\alpha} = o(1) \text{ and } d_g(x_0, y_\alpha) = O(\mu_\alpha).$$

Independently, we consider an exponential chart $(\Omega_0, \exp_{x_0}^{-1})$ centered at x_0 such that $\exp_{x_0}^{-1}(\Omega_0) = \mathbb{B}_{R_0}(0)$, $R_0 \in (0, i_g(M))$. Under the same assumptions of Theorem 2, we assume that $z_\alpha = x_0$ and we let $\tilde{Y}_\alpha = \mu_\alpha^{-1} \exp_{x_0}^{-1}(y_\alpha)$ and $\hat{g}_{0,\alpha} : X \in \mathbb{B}_{R_0}(0) \mapsto \exp_{x_0}^* g(\mu_\alpha X)$.

By compactness arguments, there exists a constant $C_{23} > 1$ such that for all $X, Y \in \mathbb{R}^n$, $\mu_\alpha |X|, \mu_\alpha |Y| < R_0$:

$$C_{23}^{-1} |X - Y| \leq d_{\hat{g}_{0,\alpha}}(X, Y) \leq C_{23} |X - Y|.$$

Then we have :

$$(67) \quad |\tilde{Y}_\alpha| = O(1) \text{ and } \mu_\alpha^{-1} \exp_{x_0}^{-1}(\mathbb{B}_{\hat{r}_\alpha}(y_\alpha)) \subseteq \mathbb{B}_{C_{23} \frac{\hat{r}_\alpha}{\mu_\alpha}}(\tilde{Y}_\alpha).$$

Using (67) and the change of variable $X = \mu_\alpha^{-1} \exp_{x_0}^{-1}(x)$, we obtain :

$$\int_{\mathbb{B}_{\hat{r}_\alpha}(y_\alpha) \cap \mathbb{B}_{R\mu_\alpha}(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g \leq \int_{\mathbb{B}_{C_{23} \frac{\hat{r}_\alpha}{\mu_\alpha}}(\tilde{Y}_\alpha)} \frac{\hat{u}_\alpha^{2^*(s)}}{d_{\hat{g}_{0,\alpha}}(X, 0)^s} dv_{\hat{g}_0}.$$

By dominated convergence Theorem, it follows that

$$\int_{\mathbb{B}_{\hat{r}_\alpha}(y_\alpha) \cap \mathbb{B}_{R\mu_\alpha}(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g = o(1).$$

Therefore, from the last relation and (63), we get (61). This ends the proof in the Case 2.2.2.

Now, we consider a family $(\Omega_\alpha, \exp_{y_\alpha}^{-1})_{\alpha>0}$ of exponential charts centered at y_α and we define on $\mathbb{B}_{R_0 \hat{r}_\alpha^{-1}}(0) \subset \mathbb{R}^n$, $R_0 \in (0, i_g(M))$ the function $\bar{u}_\alpha(X) = \hat{r}_\alpha^{\frac{n}{2}-1} u_\alpha(\exp_{y_\alpha}(\hat{r}_\alpha X))$ and the metric $\bar{g}_\alpha(X) = \exp_{y_\alpha}^* g(\hat{r}_\alpha X)$. Using the same arguments of Step 1.2, we prove that there exists $\bar{u} \in D_1^2(\mathbb{R}^n)$ such that $\bar{u}_\alpha \rightarrow \bar{u}$ weakly in $D_1^2(\mathbb{R}^n)$ as $\alpha \rightarrow +\infty$. To prove that \bar{u} is non vanishing, we need the following Lemma :

Lemma 2. *The sequence $(\bar{u}_\alpha)_{\alpha>0}$ is C^0 -bounded on any compact in \mathbb{R}^n .*

Indeed, by (59) we have that

$$(68) \quad \bar{u}_\alpha(X) \leq \left(\frac{d_g(x_0, y_\alpha)}{d_g(x_0, \exp_{y_\alpha}(\hat{r}_\alpha X))} \right)^{\frac{n}{2}-1}$$

for all $X \in \mathbb{B}_{R_0 \hat{r}_\alpha^{-1}}(0)$. Given $R > 0$, we get for all $X \in \mathbb{B}_R(0)$ that

$$d_g(x_0, \exp_{y_\alpha}(\hat{r}_\alpha X)) \geq d_g(x_0, y_\alpha) - R\hat{r}_\alpha.$$

By (68) and the last triangular inequality, we get for all $X \in \mathbb{B}_R(0)$ that

$$(69) \quad \bar{u}_\alpha(X) \leq \left(1 - R \frac{\hat{r}_\alpha}{d_g(x_0, y_\alpha)} \right)^{\frac{-2}{n-2}}.$$

By (60), we have that $d_g(x_0, y_\alpha)^{-1} \hat{r}_\alpha = o(1)$. Combining this last relation with (69), we get for all $X \in \mathbb{B}_R(0)$, that $\bar{u}_\alpha(X) \leq 1 + o(1)$ in $C^0(\mathbb{B}_R(0))$. This ends the proof of the Lemma.

Since $\bar{u}_\alpha(0) = 1$ for all $\alpha > 0$ then using the same arguments of Step 1.1 and Lemma 2, we obtain by Theorem 4.1 in Han-Lin [14] (see also Lemma 3 in the Appendix) that there exists $C_{24} > 0, r > 0$ independent of α such that $\|\bar{u}_\alpha\|_{L^2(\mathbb{B}_r(0))} \geq C_{24}$. Letting $\alpha \rightarrow +\infty$ in the last relation, we deduce that $\bar{u} \not\equiv 0$. Similarly, we prove as in Step 1.7 that $\hat{u}_\alpha \rightarrow \hat{u}$ in $C_{loc}^0(\mathbb{R}^n)$.

Coming back to (61), we write that for any $\alpha > 0$ that

$$(70) \quad \int_{\mathbb{B}_1(0)} \frac{\bar{u}_\alpha^{2^*(s)}}{d_{\bar{g}_\alpha}(X, X_{0,\alpha})^s} dv_{\bar{g}_\alpha} = \int_{\mathbb{B}_{\hat{r}_\alpha}(y_\alpha)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g = o(1) + \varepsilon_R,$$

where the function $\varepsilon_R : \mathbb{R} \rightarrow \mathbb{R}$ verifies $\lim_{R \rightarrow +\infty} \varepsilon_R = 0$. Letting $\alpha \rightarrow +\infty$ then $R \rightarrow +\infty$ in the last relation, we then get : $\int_{\mathbb{B}_1(0)} |X|^{-s} \bar{u}^{2^*(s)} dX = 0$. Contradiction since $\bar{u} \in C^0(\mathbb{B}_1(0))$ and $\bar{u}(0) = \lim_{\alpha \rightarrow 0} \bar{u}_\alpha(0) = 1$. This ends Step 2.2. \square

Step 2.3: Here goes the final argument (we adapt the one in Druet [10] and in Hebey [17] to our case). We fix $\rho \in (0, i_g(M))$ sufficiently small. We consider a smooth cut-off function η on M such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $\mathbb{B}_\rho(x_0)$ and $\eta \equiv 0$ on $M \setminus \mathbb{B}_{2\rho}(x_0)$. We define the function η_0 on $\mathbb{B}_{2\rho}(x_0)$ by $\eta_0 = \eta \circ \exp_{x_0}^{-1}$. We let $dx = (\exp_{x_0}^{-1})^* dX$ and $\tilde{\delta}_0 = (\exp_{x_0}^{-1})^* \delta$. We consider two constants, $C_{25}, C_{26} > 0$, independents of α such that $|\nabla \eta_0|_g \leq C_{25}$ et $|\Delta_g \eta_0|_g \leq C_{26}$. The sharp Euclidean Hardy-Sobolev inequality gives for all $\alpha > 0$ that

$$\left(\int_{\mathbb{R}^n} \frac{|\eta(u_\alpha \circ \exp_{x_0})|^{2^*(s)}}{|X|^s} dX \right)^{\frac{2}{2^*(s)}} \leq K(n, s) \int_{\mathbb{R}^n} |\nabla(\eta(u_\alpha \circ \exp_{x_0}))|_\delta^2 dX.$$

This implies that for all $\alpha > 0$ we have :

$$(71) \quad \left(\int_M \frac{|\eta_0 u_\alpha|^{2^*(s)}}{d_{\tilde{\delta}_0}^s(x, x_0)} dx \right)^{\frac{2}{2^*(s)}} \leq K(n, s) \int_M |\nabla(\eta_0 u_\alpha)|_{\tilde{\delta}_0}^2 dx.$$

In order to get a contradiction, we estimate the RHS (respectively the LHS) of the Equation (71), by comparing the L^2 - norm of $|\nabla(\eta_0 u_\alpha)|_{\tilde{\delta}_0}$ (resp. the $L^{2^*(s)}$ - norm of $\eta_0 u_\alpha$) with respect to $\tilde{\delta}_0$ with the L^2 - norm of $|\nabla u_\alpha|_g$ (resp. the $L^{2^*(s)}$ - norm of u_α) with respect to g . We let $r_0(x) = d_g(x, x_0)$ be the geodesic distance to x_0 . Cartan's expansion of the metric g (see [21]) in the exponential chart $(\mathbb{B}_{2\rho}(x_0), \exp_{x_0}^{-1})$ yields

$$(72) \quad \begin{aligned} \int_M |\nabla(\eta_0 u_\alpha)|_{\tilde{\delta}_0}^2 dx &= \int_M (1 + C_{27} r_0^2(x)) |\nabla(\eta_0 u_\alpha)|_g^2 (1 + C_{28} r_0^2(x)) dv_g \\ &\leq \int_M |\nabla(\eta_0 u_\alpha)|_g^2 dv_g + C_{29} \int_M r_0^2(x) |\nabla(\eta_0 u_\alpha)|_g^2 dv_g \\ &\leq \int_M |\nabla(\eta_0 u_\alpha)|_g^2 dv_g + \int_M r_0^2 \eta_0^2 |\nabla u_\alpha|_g^2 dv_g \\ &\quad + C_{30} \int_{M \setminus \mathbb{B}_\rho(x_0)} u_\alpha^2 dv_g, \end{aligned}$$

where $C_i > 0, i = 27, \dots, 30$ are independent of α . Independently, we get by integrating by parts that

$$(73) \quad \int_M |\nabla(\eta_0 u_\alpha)|_g^2 dv_g = \int_M \eta_0^2 |\nabla u_\alpha|_g^2 dv_g + \int_M \eta_0 \Delta_g \eta u_\alpha^2 dv_g$$

$$(74) \quad \leq \int_M |\nabla u_\alpha|_g^2 dv_g + C_{26} \int_M u_\alpha^2 dv_g$$

We let now $f_0 := \eta_0^2 r_0^2$ which is a smooth function. So that

$$(75) \quad \int_M \eta_0^2 r_0^2 |\nabla u_\alpha|_g^2 dv_g = \int_M (\nabla(f_0 u_\alpha) - u_\alpha \nabla f_0, \nabla u_\alpha)_g dv_g.$$

Multiplying equation (57) by $f_0 u_\alpha$ then integrating by parts over M , we get :

$$(76) \quad \begin{aligned} \int_M (\nabla(f_0 u_\alpha), \nabla u_\alpha)_g dv_g &= \int_M (\Delta_g u_\alpha) f_0 u_\alpha dv_g \\ &\leq \lambda_\alpha \int_M \frac{f_0 u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g. \end{aligned}$$

By Step 2.2, there exists a constant $C_{31} > 0$ independent of α such that we have for all $x \in M$:

$$(77) \quad u_\alpha^{2^*(s)}(x) d_g(x, x_0)^{2-s} \leq C_{31} u_\alpha^2(x).$$

Since $\lambda_\alpha \in (0, K(n, s)^{-1})$ then by (76) et (77), we get :

$$(78) \quad \int_M (\nabla(f_0 u_\alpha), \nabla u_\alpha)_g dv_g \leq C_{32} \int_M u_\alpha^2(x) dv_g,$$

where $C_{32} > 0$ is a constant independent of α . Integrating by parts gives

$$(79) \quad \int_M (\nabla f_0, \nabla u_\alpha)_g u_\alpha dv_g = \int_M \frac{1}{2} u_\alpha \Delta_g(f_0) dv_g \leq C_{33} \int_M u_\alpha^2 dv_g,$$

where $C_{33} > 0$ is a constant independent of α . Plugging (79) and (78) into (75), we get that

$$(80) \quad \int_M \eta_0^2 r_0^2 |\nabla u_\alpha|_g^2 dv_g \leq C_{34} \int_M u_\alpha^2 dv_g,$$

where $C_{34} > 0$ is a constant independent of α . Therefore (72) yields

$$(81) \quad \int_M |\nabla(\eta_0 u_\alpha)|_{\delta_0}^2 dx \leq \int_M |\nabla u_\alpha|_g^2 dv_g + C_{35} \int_M u_\alpha^2 dv_g,$$

where the constants $C_{35} > 0$ is independent of α .

On the other hand, we know by Gauß's Lemma that $d_{\delta_0}(x, x_0) = d_g(x, x_0) = |\exp_{x_0}^{-1}(x)|$. Writing that $dx = dv_g + (1 - \sqrt{\det(g)})dx$ and thanks to Cartan's expansion (see Lee-Parker [21]), we obtain

$$(82) \quad \int_M \frac{|\eta_0 u_\alpha|^{2^*(s)}}{d_{\delta_0}(x, x_0)^s} dx \geq \int_M \frac{|\eta_0 u_\alpha|^{2^*(s)}}{d_g(x, x_0)^s} dv_g - \int_M \frac{|\eta_0 u_\alpha|^{2^*(s)}}{d_g(x, x_0)^s} C_{36} r_0^2(x) dx,$$

where the constants $C_{36} > 0$ is independent of α . With (77) and (58), we have that

$$(83) \quad \begin{aligned} \int_M \frac{r_0^2(x) |\eta_0 u_\alpha|^{2^*(s)}}{d_g(x, x_0)^s} dx &\leq C_{37} \int_M (u_\alpha(x) d_g(x, x_0)^{\frac{n}{2}-1})^{\frac{2(2-s)}{n-2}} u_\alpha^2(x) dv_g \\ &\leq C_{38} \int_M u_\alpha^2(x) dv_g, \end{aligned}$$

where $C_{37}, C_{38} > 0$ is a constant independent of α . Since we have for all $\alpha > 0$, we have that $\int_M d_g(x, x_0)^{-s} |\eta_0 u_\alpha|^{2^*(s)} dv_g \leq 1$ and, up to a subsequence, that $\int_M u_\alpha^2 dv_g = o(1)$, it then follows with (82) and (83) that

$$(84) \quad \begin{aligned} \left(\int_M \frac{|\eta_0 u_\alpha|^{2^*(s)}}{d_g(x, x_0)^s} dx \right)^{\frac{2}{2^*(s)}} &\geq \left(\int_M \frac{|\eta_0 u_\alpha|^{2^*(s)}}{d_g(x, x_0)^s} dv_g - C_{38} \int_M u_\alpha^2(x) dv_g \right)^{\frac{2}{2^*(s)}} \\ &\geq \int_M \frac{|\eta_0 u_\alpha|^{2^*(s)}}{d_g(x, x_0)^s} dv_g - C_{38} \int_M u_\alpha^2(x) dv_g. \end{aligned}$$

Now the definition of η_0 gives that

$$(85) \quad \int_M \frac{|\eta_0 u_\alpha|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \geq \int_M \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g - \int_{M \setminus \mathbb{B}_\rho(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g.$$

Since $u_\alpha \rightarrow 0$ as $\alpha \rightarrow +\infty$ in $C_{loc}^0(M \setminus \{x_0\})$ (Proposition 1), there exists a constant $C_{39} > 0$ independent of α such that

$$\begin{aligned} \int_{M \setminus \mathbb{B}_\rho(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g &\leq \sup_{x \in M \setminus \mathbb{B}_\rho(x_0)} \left(\frac{u_\alpha^{2^*(s)-2}}{d_g(x, x_0)^s} \right) \int_{M \setminus \mathbb{B}_\rho(x_0)} u_\alpha^2 dv_g \\ (86) \qquad \qquad \qquad &\leq C_{39} \int_M u_\alpha^2 dv_g. \end{aligned}$$

Combining (57), (85) and (86) yields

$$\begin{aligned} \int_M \frac{|\eta_0 u_\alpha|^{2^*(s)}}{d_g(x, x_0)^s} dv_g &\geq \frac{1}{\lambda_\alpha} \left(\int_M |\nabla u_\alpha|_g^2 dv_g + \alpha \int_M u_\alpha^2 dv_g \right) \\ (87) \qquad \qquad \qquad &\quad - C_{39} \int_M u_\alpha^2 dv_g. \end{aligned}$$

Plugging (87) into (84), and using that $\lambda_\alpha < K(n, s)^{-1}$, we obtain that

$$(88) \qquad \left(\int_M \frac{|\eta_0 u_\alpha|^{2^*(s)}}{d_g(x, x_0)^s} dx \right)^{\frac{2}{2^*(s)}} \geq K(n, s) \int_M |\nabla u_\alpha|_g^2 dv_g + (\alpha K(n, s) - C_{40}) \int_M u_\alpha^2 dv_g,$$

where $C_{40} > 0$ is a constant independent of α . Combining (81), (88) with (71), we then get :

$$(89) \qquad (C_{41} - \alpha) \int_M u_\alpha^2 dv_g \geq 0,$$

where $C_{41} > 0$ is a constant independent of α . Contradiction since $\alpha \rightarrow +\infty$. This ends the proof of Theorem 1.

Proof of (7). We write $B := B_0(M, g, s, x_0)$ for simplicity. It follows from (6) and the definition (9) of I_α that

$$(90) \qquad K(n, s)^{-1} \leq \inf_{u \in H_1^2(M) \setminus \{0\}} I_{K(n, s)^{-1}B}(u)$$

We define the test-function sequence $(u_\epsilon)_{\epsilon > 0}$ by

$$u_\epsilon(x) = \left(\frac{\epsilon^{1-\frac{s}{2}}}{\epsilon^{2-s} + d_g(x, x_0)^{2-s}} \right)^{\frac{n-2}{2-s}} \quad \text{for } n \geq 4$$

for all $\epsilon > 0$ and $x \in M$. When $n = 3$, we let G_{x_0} be the Green's function for the coercive operator $\Delta_g + K(3, s)^{-1}B$, and we define $\beta_{x_0} := G_{x_0} - \eta d_g(\cdot, x_0)^{-1}$ where η is a cut-off function around x_0 . Note that $\beta_{x_0} \in C^0(M)$, and the mass of G_{x_0} is $\beta_{x_0}(x_0)$. We define for any $\epsilon > 0$

$$u_\epsilon(x) = \eta(x) \left(\frac{\epsilon^{1-\frac{s}{2}}}{\epsilon^{2-s} + d_g(x, x_0)^{2-s}} \right)^{\frac{n-2}{2-s}} + \sqrt{\epsilon} \beta(x) \quad \text{for } n = 3$$

for all $x \in M$. It follows from [20] that

$$I_{K(n, s)^{-1}B}(u_\epsilon) = K(n, s)^{-1} + \gamma_n \Omega_n(x_0) \theta_\epsilon + o(\theta_\epsilon)$$

when $\epsilon \rightarrow 0$, where $\gamma_n > 0$ for all $n \geq 3$, $\theta_\epsilon := \epsilon^2$ if $n \geq 5$, $\theta_\epsilon := \epsilon^2 \ln(\epsilon^{-1})$ if $n = 4$, $\theta_\epsilon := \epsilon$ if $n = 3$, and

$$\Omega_n(x_0) := K(n, s)^{-1}B - \frac{(n-2)(6-s)}{12(2n-2-s)} \text{Scal}_g(x_0) \quad \text{if } n \geq 4; \quad \Omega_3(x_0) := -\beta_{x_0}(x_0).$$

It then follows from (90) that $\Omega_n(M, g, s, x_0) \geq 0$. This proves (7).

3. APPENDIX

Following arguments as in Han and Lin [14] (see Theorem 4.1), we have that

Lemma 3. *Let $\mathbb{B}_2(0)$ be the ball in \mathbb{R}^n of center 0 and radius 2, \tilde{g} be a Riemannian on $\mathbb{B}_2(0)$ and let $A = A(\tilde{g}) > 0$ be such that for all $\phi \in C_c^\infty(\mathbb{B}_2(0))$, we have :*

$$\|\phi\|_{L^{2^*}_{\tilde{g}}(\mathbb{B}_1(0))} \leq A \|\nabla \phi\|_{L^2_{\tilde{g}}(\mathbb{B}_1(0))},$$

where $L^2_{\tilde{g}}$ is the Lebesgue space of $(\mathbb{B}_1(0), dv_{\tilde{g}})$. We consider $u \in H^2_1(\mathbb{B}_1(0), \tilde{g})$, $u \geq 0$ a.e. such that we have $\Delta_{\tilde{g}} u \leq fu$, on $H^2_{1,0}(\mathbb{B}_1(0), \tilde{g})$ and $\int_{\mathbb{B}_1(0)} |f|^r dv_{\tilde{g}} \leq k$ with $r > \frac{n}{2}$ and $k > 0$ is a constant depending of $(M, g), f, r$. Then $u \in L^\infty_{loc}(\mathbb{B}_1(0))$. Moreover, for all $p > 0$, there exists a constant $C_{42} = C(n, p, r, \tilde{g}, k)$ such that for all $\theta \in]0, 1[$ we have :

$$\sup_{\mathbb{B}_\theta(0)} u \leq C_{42} \frac{1}{(1 - \theta)^{\frac{n}{p}}} \|u\|_{L^p_{\tilde{g}}(\mathbb{B}_1(0))}.$$

We use another version of this lemma adapted for compact Riemannian manifolds.

Lemma 4. *Let (M, g) be a compact Riemannian Manifold. We consider $u \in H^2_1(M)$, $u \geq 0$. We fix an open domain Ω of M and assume that u verifies*

$$\begin{cases} \Delta_g u \leq fu, & \text{on } \Omega \text{ in the sense of distributions,} \\ \int_\Omega |f|^r dv_g \leq C_{43}, & r > \frac{n}{2}, \end{cases}$$

with $C_{43} = C_{43}(M, g, f, r)$ then for all $\omega \subset \subset \Omega$ and all $p > 0$, there exists $C_{44} = C_{44}(M, g, C_{43}, p, r, \Omega, \omega) > 0$ (independent of u) such that

$$\|u\|_{L^\infty(\omega)} \leq C_{44} \|u\|_{L^p(\Omega)}.$$

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